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A Characterization of the Partial Geometry $T_2^*(K)$

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In this paper we characterize the partial geometry $T_2^*(K)$ embedded in AG(3, q) as a net-inducible partial geometry. This characterization is closely related to the characterization theorem of the generalized quadrangle $T_2^*(O)$ in [8].

1. INTRODUCTION

1.1. DEFINITIONS. A finite partial geometry $S = (P, B, \mathrm{I})$ is an incidence structure in which P and B are sets of objects called points and lines, respectively, with a symmetric incidence relation satisfying the following axioms:
(a) each point is incident with $t + 1$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;
(b) each line is incident with $s + 1$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point;
(c) if x is a point and L is a line, such that $x \not\mathrm{I}\, L$, then there exist exactly $\alpha(\alpha \geqslant 1)$ points $x_1, \ldots, x_\alpha$ and α lines $L_1, \ldots, L_\alpha$, such that $x \mathrm{I} L_i \mathrm{I} x_i \mathrm{I} L$, $i = 1, 2, \ldots, \alpha$.

The numbers s, t and α are called the parameters of the partial geometry. This incidence structure was introduced by R. C. Bose in 1963 [2].

We denote collinear points x and y (respectively concurrent lines L, M) by $x \sim y$ (resp. $L \sim M$).

For $x \in P$, put $x^\perp = \{y \in P \| y \sim x\}$. Remark that $x \in x^\perp$. The trace of a pair of distinct points x, y is defined to be the set $x^\perp \cap y^\perp$ and is denoted by $\{x, y\}^\perp$. There holds $|\{x, y\}^\perp| = s + 1 + t(\alpha - 1)$ or $(t + 1)\alpha$ according as $x \sim y$ or $x \not\sim y$. More generally, for $A \subset P$, we define the set $A^\perp$ as $\cap\{x^\perp \| x \in A\}$. For $x \neq y$, the span of the pair (x, y) is the set $\{x, y\}^{\perp\perp} = \{u \in P \| u \in z^\perp, \forall z \in \{x, y\}^\perp\}$. A triad of points is a set of three pairwise non-collinear points (x, y, z). These definitions can be dualized.

1.2. REMARKS (a) If $S = (P, B, \mathrm{I})$ is a partial geometry with parameters s, t and α, then the dual incidence structure $\bar{S} = (\bar{P}, \bar{B}, \bar{\mathrm{I}})$, $\bar{P} = B$, $\bar{B} = P$, $\bar{\mathrm{I}} = \mathrm{I}$, is a partial geometry with parameters $\bar{t} = s$, $\bar{s} = t$ and $\bar{\alpha} = \alpha$.
(b) There holds $|P| = v = (s + 1)\dfrac{(st + \alpha)}{\alpha}$ and $|B| = b = (t + 1)\dfrac{(st + \alpha)}{\alpha}$, hence $\alpha | st(s + 1)$ and $\alpha | st(t + 1)$.
(c) The point graph of a partial geometry is strongly regular with parameters v, $k = s(t + 1)$, $\lambda = s - 1 + t(\alpha - 1)$, $\mu = (t + 1)\alpha$ [2]. The integrality condition for strongly regular graphs yields $\alpha(s + t + 1 - \alpha) | st(t + 1)(s + 1)$, while the Krein condition yields $(s + 1 - 2\alpha)t \leqslant (s - 1)(s + 1 - \alpha)^2$ (and dually).

1.3. THE FOUR CLASSES OF PARTIAL GEOMETRIES. The partial geometries can be divided into four (non-disjoint) classes:
(a) the generalized quadrangles: $\alpha = 1$ [12];
(b) the $2 - (v, s + 1, 1)$ designs: $\alpha = s + 1$; and their duals: $\alpha = t + 1$;
(c) the nets of order $s + 1$ and degree $t + 1$: $\alpha = $ t; and their duals: $\alpha = s$;
(d) the 'proper' partial geometries: $1 < \alpha < \min(s, t)$.

1.4. THE PARTIAL GEOMETRY $T_2^*(K)$ [4]. A maximal arc K of degree d in a finite projective plane of order q (not necessarily desarguesian) is a (maximal) set of $qd - q + d$

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points of the plane such that any line of the plane intersects K in 0 or d points. If K is a proper subset of the plane, one can easily prove that d has to divide q [10].

Let K be a maximal arc of degree d in the projective plane PG(2, q) over GF(q) ($q = p^h$, p a prime). We define an incidence structure $S = (P, B, \mathrm{I})$ as follows. Let PG(2, q) be embedded as a plane H in PG(3, q). The points of S are the points of PG(3, q)$\backslash H$; the lines of S are the lines of PG(3, q) which are not contained in H and meet K (necessarily in a unique point). The incidence is that of PG(3, q). Then S is a partial geometry with parameters $t = qd - q + d - 1$, $s = q - 1$, $\alpha = d - 1$ and is denoted by $T_2^*(K)$.

The partial geometry $T_2^*(K)$ using an arc of degree 2^m in PG(2, 2^h) has parameters $s = 2^h - 1$, $t = (2^h + 1)(2^m - 1)$, $\alpha = 2^m - 1$. This is a generalized quadrangle iff $m = 1$; i.e. K is a complete oval [10]. These generalized quadrangles corresponding with complete ovals were first discovered by R. W. Ahrens and G. Szekeres [1] and independently by M. Hall Jr [9].

2. α-Regularity in Partial Geometries

In [5] α-regularity in a partial geometry S with parameters s, t, α ($t + 1 \neq \alpha \neq s + 1$) is defined in the following way. Let L and M be two non-concurrent lines, then one can prove that $\{L, M\}^{\perp\perp}$ is a set of pairwise non-concurrent lines. Hence $|\{L, M\}^{\perp\perp}| \leqslant s + 1$, if equality holds the pair (L, M) is called α-regular. If $\alpha = 1$, (L, M) is called regular [12]. It is proved in [5] that a partial geometry S with parameters t, s, α ($s + 1 \neq \alpha \neq t + 1$) containing an α-regular pair of lines, satisfies $(s + 1)\alpha \leqslant t + 1$.

If (L, M) is an α-regular pair of lines in S we can construct a net $\mathcal{N}(L, M)$ of order $s + 1$ and degree $\alpha + 1$ as follows: the lineset is the set $\{L, M\}^{\perp} \cup \{L, M\}^{\perp\perp}$, the pointset is the set of points of S incident with these lines [5]. Any line of S which has at least two points in common with $\mathcal{N}(L, M)$ is a line of this net. Note that any pair (N, N') of non-concurrent lines in $\mathcal{N}(L, M)$ is also α-regular and that $\mathcal{N}(L, M) = \mathcal{N}(N, N')$.

3. Net-Inducible Partial Geometries

Let $S = (P, B, \mathrm{I})$ be a partial geometry with parameters s, t, α, ($s + 1 \neq \alpha \neq t + 1$), $\alpha > 1$. In the sequel we assume, without loss of generality, that the lines of B are subsets of the pointset P and hence the incidence relation is the natural one.

3.1. Definitions. We call two lines $L, L' \in B$ parallel (notation $L \| L'$) iff $L = L'$ or $L \cap L' = \emptyset$ and there exist two lines $M_1, M_2 \in B$ such that $M_1 \cap M_2 = \{x\}$, $x \in P$, $x \notin L \cup L'$ and $M_i \cap L \neq \emptyset$, $M_i \cap L' \neq \emptyset$, $i = 1, 2$.

Let L, M be different lines, $L \cap M = \{x\}$ or $L \| M$. We define for any point p, $p \notin L \cup M$, the integer $\beta(p)$ w.r.t. L and M, as $|\{K \in B \| p \in K, K \in \{L, M\}^{\perp}\}|$. Any line $K \in \{L, M\}^{\perp}$ for which $x \notin K$ is called a transversal of L and M.

3.2. Axiom (A). *Let L and M be two different intersecting or parallel lines of S. For any point p on a transversal of L and M, $p \notin L \cup M$, there holds $\beta(p) = \alpha - |L \cap M|$ with respect to L and M.*

In the sequel we will always assume that the partial geometry S satisfies Axiom (A).

3.3. Remark Let p be a point on a transversal of two different concurrent lines L and M, $p \notin L \cup M$. Then it is easy to check that the unique line L' (resp. M') containing p, for which $L' \cap L \neq \emptyset$ but $L' \cap M = \emptyset$ (respectively $M \cap M' \neq \emptyset$ but $M' \cap L = \emptyset$) satisfies $L' \| M$ (resp. $M' \| L$).

3.4. THEOREM. *If L and M are two different parallel lines, then (L, M) is an α-regular pair.*

PROOF. By definition of parallelism there exist two lines L_1, $L_2 \in B$ such that $L_1 \cap L_2 = \{x\}$, $x \notin L \cup M$, $L_i \cap L \neq \emptyset$, $L_i \cap M \neq \emptyset$, $i = 1, 2$. In view of the definition of α-regularity we have to prove $|\{L, M\}^{\perp\perp}| = s + 1$. Let $\{L, M\}^{\perp} = \{N_{11}, \ldots, N_{1\alpha}, N_{21}, \ldots, N_{2\alpha}, \ldots, N_{s+1,1}, \ldots, N_{s+1,\alpha}\}$ with $N_{11} = L_1$, $N_{s+1,\alpha} = L_2$ and $\bigcap_{j=1}^{\alpha} N_{ij} = \{x_i\}$ with $x_i \in L$, $1 \leqslant i \leqslant s + 1$. Applying Axiom (A) and in view of Remark 3.3. we can construct through every point of $L_1 \backslash \{x, x_1, L_1 \cap M\}$ a unique parallel to L and M. Indeed, let $y \in L_1$, $x \neq y \neq x_1$, $y \notin M$. As L_1 is a transversal of L and L_2 and $L \cap L_2 \neq \emptyset$, there follows by 3.3. that y is on a unique parallel N to L and $N \cap L_2 \neq \emptyset$. Clearly we also have $N \| M$. We show that $N \in \{L, M\}^{\perp\perp}$. By Axiom (A) there holds that each of the points $N \cap L_1$, respectively $N \cap L_2$, is contained in α transversals of L and M. Now we consider a point $z \in N$, $z \notin L_1 \cup L_2$. Since z is on the transversal N of L_1 and L_2, and $L_1 \cap L_2 \neq \emptyset$, it follows from Axiom (A) and 3.3. that there exists a unique parallel L_2' to L_2 through z and $L_1 \cap L_2' = \{z'\}$. Now z' is on the transversal L_1 of N and L and applying Axiom (A) we obtain that $L_2' \cap L \neq \emptyset$ and analogously $L_2' \cap M \neq \emptyset$. Hence z is on a transversal of L and M, and so, z is on a α transversals of L and M. So we obtain that each of the $s + 1$ points of N is on α transversals of L and M, and of course they are all different. Since $|\{L, M\}^{\perp}| = \alpha(s + 1)$ we may conclude that $N \in \{L, M\}^{\perp\perp}$. This holds for any point y, different from x and not incident with L or M. Moreover x is also incident with a unique line parallel to L, because the same construction as before can be given starting with the transversals L_1, L_2' of L and M, $L_1 \cap L_2' = \{z'\}$. Hence any point of L_1 is on a unique parallel line to L, which belongs to $\{L, M\}^{\perp\perp}$. There follows that $|\{L, M\}^{\perp\perp}| = s + 1$.

3.5. COROLLARY. As any two different parallel lines are α-regular they define a net of order $s + 1$ and degree $\alpha + 1$ [5].

Now, let L and M be two intersecting lines with $L \cap M = \{x\}$. Let y be a point of M different from x. As $\alpha \neq 1$, y is incident with at least one transversal of L and M. Any point z on this transversal ($z \notin L \cup M$) is by Remark 3.3 incident with a uniquely defined line N, $L \| N$ and $N \cap M \neq \emptyset$. As M has at least two points in common with $\mathcal{N}(L, N)$, M is a line of this net. Moreover, each of the α lines through x and concurrent to N belongs to $\mathcal{N}(L, N)$. Hence, L and M define uniquely this net.

Moreover, if L is a line and y is a point not on L, then there exists at least one line M through y intersecting L and hence L and y define uniquely a net. Consequently there is a unique parallel through y to L in this net.

From now on we will denote this net by $\mathcal{N}(L, M)$ for any two intersecting or distinct parallel lines or by $\mathcal{N}(L, y)$ for any non-incident point line pair (y, L), and we will say that L and M or L and y generate this net, and therefore we will call them generated nets.

3.6. DEFINITION. Any proper partial geometry S satisfying Axiom (A) will be called a net-inducible partial geometry.

3.7. THEOREM If $S = (P, B, \mathrm{I})$ is a net-inducible partial geometry with parameters s, t, α, then

(a) every two non-collinear points are contained in exactly $(t + 1)/(\alpha + 1)$ generated nets;
(b) every line of S is contained in t/α generated nets;
(c) $t \geqslant \alpha(s + 2)$.

PROOF. By counting arguments (a) and (b) are immediately proved. For (c), we consider a net $\mathcal{N}(L, M)$ and a point y, which is not contained in this net. $\mathcal{N}(L, M)$ is a net of order

$s + 1$ and degree $\alpha + 1$, hence y is collinear with $\alpha(s + 1)$ points of $\mathcal{N}(L, M)$. Moreover y is incident with $\alpha + 1$ lines which are parallel to $\alpha + 1$ lines of $\mathcal{N}(L, M)$ through a general point. Hence $t \geqslant \alpha(s + 2)$.

3.8. THEOREM. If $S = (P, B, \mathrm{I})$ is a net-inducible partial geometry with parameters s, t, α and $t = \alpha(s + 2)$, then parallelism between lines is an equivalence relation.

PROOF. Clearly this relation is reflexive and symmetric. We prove now the transitivity. Let L, M and N be three different lines such that $L \parallel M$ and $M \parallel N$. If $\mathcal{N}(L, M) = \mathcal{N}(M, N)$, then $L \parallel N$. Suppose now that $\mathcal{N}(L, M) \neq \mathcal{N}(M, N)$. Let x be an arbitrary point of N and let N' be the line incident with x and parallel to L in the generated net $\mathcal{N}(L, x)$. Assume $N \neq N'$. Let y be an arbitrary point of $\mathcal{N}(L, M)$. Then x is incident with α lines which are parallel to the α transversals of L and M through y. Clearly these lines are different from N and N' and this implies that $t + 1 \geqslant \alpha(s + 1) + \alpha + 2$, a contradiction. Consequently $N = N'$ and parallelism between lines is an equivalence relation.

3.9. PARALLELISM BETWEEN GENERATED NETS. In the sequel we always assume $t = \alpha(s + 2)$. Let $\mathcal{N}$ be a generated net in the partial geometry S and let y be an arbitrary point of S, which is not contained in $\mathcal{N}$. The $\alpha + 1$ lines through y which have no point in common with $\mathcal{N}$ are contained in a generated net $\mathcal{N}'$ and obviously $\mathcal{N} \cap \mathcal{N}' = \varnothing$.

Any two generated nets will be called parallel (and we will write as for lines $\mathcal{N} \parallel \mathcal{N}'$), iff $\mathcal{N} = \mathcal{N}'$ or $\mathcal{N} \cap \mathcal{N}' = \varnothing$. Clearly two parallel nets define the same $\alpha + 1$ parallel classes of lines. It follows easily that parallelism between generated nets is an equivalence relation and from the preceding paragraph follows that each parallel class of nets partitions the set of points P.

We can now define the following design $S' = (P', B', \mathrm{I}')$ with P' the set of parallel classes of lines and with B' the set of parallel classes of generated nets, the incidence I' being the natural one. It is clear that S' is a $2 - (\alpha(s + 2) + 1, \alpha + 1, 1)$ design.

3.10. REMARK Let $\mathcal{N}$ and $\mathcal{N}'$ be two different generated nets such that $\mathcal{N} \cap \mathcal{N}' \neq \varnothing$ and there exist at least two different lines L, L' $L \in \mathcal{N}$, $L' \in \mathcal{N}'$ such that $L \parallel L'$, then $\mathcal{N}$ and $\mathcal{N}'$ intersect in a line parallel to L (and L').

If L is a line and $\mathcal{N}$ is a generated net such that $L \cap \mathcal{N} = \varnothing$, then there exists at least one line M (and hence $s + 1$ lines) of $\mathcal{N}$, such that $L \parallel M$.

If $\mathcal{N} \cap \mathcal{N}' \neq \varnothing$ and $\mathcal{N} \cap \mathcal{N}'$ is not a line then $\mathcal{N} \cap \mathcal{N}'$ is clearly a set of $s + 1$ pairwise non-collinear points.

4. $(0, \alpha + 1)$-SETS AND PARALLELISM

4.1. DEFINITION. Let $K \subset P$, $K \neq \varnothing$, then K is called a $(0, \alpha + 1)$-set iff K is a set of pairwise non-collinear points such that $|x^{\perp} \cap K| \in \{0, \alpha + 1\}$, $\forall x \in P \backslash K$. This definition is a generalization of the one given in [6].

4.2. EXAMPLE. Consider the partial geometry $T_2^*(K)$ described in 1.4. Let L be a line of PG$(3, q)$ which is not contained in the plane H and which does not contain a point of K. It is easy to check that the set $L' = L \backslash H$ of order q is a $(0, \alpha + 1)$-set of $T_2^*(K)$.

4.3. THEOREM. If $S = (P, B, \mathrm{I})$ is a net-inducible partial geometry such that each triad (x, y, z) of points of S is contained in exactly 0, 1 or $(t + 1)/(\alpha + 1)$ (> 1) generated nets of order $s + 1$ and degree $\alpha + 1$, then we can construct through each pair of non-collinear points of S a $(0, \alpha + 1)$-set.

PROOF. In view of 3.7. each pair of non-collinear points x, y of S is contained in $(t + 1)/(\alpha + 1)$ nets. Consider two different nets $\mathcal{N}$ and $\mathcal{N}'$ containing x and y and a point $z \in \mathcal{N} \cap \mathcal{N}'$ with $x \neq z \neq y$. By hypothesis the triad (x, y, z) is contained in $(t + 1)/(\alpha + 1)$ nets and hence all nets through x and y do contain z. All these nets contain the pointset $\mathcal{N} \cap \mathcal{N}'$. Now we check that this pointset is a $(0, \alpha + 1)$-set. The $(t + 1)/(\alpha + 1)$ nets through x and y contain

$$\frac{t+1}{\alpha+1}[(s + 1)^2 - (s + 1)] + s + 1 = (s + 1)[s(t + 1)/(\alpha + 1) + 1]$$

points of S. Each point of $\mathcal{N} \cap \mathcal{N}'$ is contained in $t + 1$ lines each of which lies in such a net. Consider an arbitrary point w of S, not contained in $\mathcal{N} \cap \mathcal{N}'$. If w belongs to one of the generated nets through x and y then the $\alpha + 1$ lines of this net through w, all intersect $\mathcal{N} \cap \mathcal{N}'$, i.e. w is collinear with $\alpha + 1$ points of $\mathcal{N} \cap \mathcal{N}'$. If w does not belong to such a generated net, w is never collinear with a point of $\mathcal{N} \cap \mathcal{N}'$.

4.4. THE DESIGN S^α In the sequel we always assume that $S = (P, B, \mathrm{I})$ is a net-inducible partial geometry such that each triad (x, y, z) of points of S is contained in exactly 0, 1 or $(t + 1)/(\alpha + 1)$ generated nets and that $t = \alpha(s + 2)$. The set of $(0, \alpha + 1)$-sets constructed through each pair of non-collinear points is denoted by B^α.

Now we define the incidence structure $S^\alpha = (P, B \cup B^\alpha, \in)$. It is obvious that S^α is a $2\text{-}((s + 1)^3, s + 1, 1)$ design. A block of S^α will either be a line or an element of B^α.

4.5. THEOREM. The substructure of S^α generated by two parallel lines L, $M \in B$ is an affine plane of order $s + 1$.

PROOF. Consider the generated net $\mathcal{N}(L, M)$. Two non-collinear points of $\mathcal{N}(L, M)$ are contained in a unique element K of B^α and from the proof of 4.3. it follows that all points of K belong to $\mathcal{N}(L, M)$. Now it is easy to verify that the substructure $S^* = (P^*, B^*, \in)$ where P^* is the set of points of $\mathcal{N}(L, M)$, $B^* = \{L, M\}^\perp \cup \{L, M\}^{\perp\perp} \cup B^{\alpha *}$ with $B^{\alpha *}$ the set of all $(0, \alpha + 1)$-sets of B^α containing at least two points of P^* is a $2\text{-}((s + 1)^2, s + 1, 1)$ design, i.e. an affine plane of order $s + 1$.

4.6. COROLLARY. The substructure of S^α generated by a point $x \in P^\alpha$ and a block $L \in B$, $x \notin L$ is an affine plane of order $s + 1$.

4.7. THEOREM. The substructure of S^α generated by a point x and a block $L \in B^\alpha$, $x \notin L$ and $|x^\perp \cap L| = \alpha + 1$ is an affine plane of order $s + 1$.

PROOF. Consider a line M of the partial geometry such that $x \in M$ and $M \cap L \neq \emptyset$ and consider a point $y \in L$, $y \notin M$. By 4.6. the substructure of S^α generated by y and M is an affine plane of order $s + 1$. Since this plane contains x and L it is also the substructure of S^α generated by x and L.

4.8. DEFINITION. In the sequel we call an affine plane arising from a generated net of order $s + 1$ and degree $\alpha + 1$, a 'generated affine plane'. Let $\pi(\mathcal{N})$ and $\pi(\mathcal{N}')$ be two such affine planes containing, respectively, the nets $\mathcal{N}$, $\mathcal{N}'$. We call $\pi(\mathcal{N})$ and $\pi(\mathcal{N}')$ parallel [notation $\pi(\mathcal{N}) \parallel \pi(\mathcal{N}')$] iff the nets $\mathcal{N}$ and $\mathcal{N}'$ are. Clearly this relation is an equivalence relation and defines a partition in the set of generated affine planes. Each parallel class defines a partition of the pointset P.

4.9. THEOREM. *Let π, π', π'', π''' be generated affine planes. Then there holds*:
(a) *if $\pi \nparallel \pi'$, then π and π' have a common block*;

(b) *if* $\pi \parallel \pi'$, $\pi \neq \pi'$, $\pi \nparallel \pi''$, *then the common blocks of* π, π'' *and* π', π'' *are both in* B *or* B^α;
(c) *if* $\pi \nparallel \pi''$, $\pi \parallel \pi'$, $\pi'' \parallel \pi'''$, *and* $\pi \cap \pi'' = L$, $\pi' \cap \pi''' = L'$, $L \subset \Omega$ *with* $\pi \neq \Omega \neq \pi''$, *then* $L' \subset \Omega'$ *with* $\Omega \parallel \Omega'$, *and* Ω, Ω', *generated affine planes.*

PROOF. Since the proof is completely analogous as the one of theorem 4.9.3. in [8], we omit it.

4.10. PARALLELISM IN B^α. We now define parallelism in B^α. If $L, M \in B^\alpha$ then L and M are called parallel ($L \parallel M$) iff there exist generated affine planes π, π', Ω, Ω' with $\pi \cap \Omega = L$, $\pi' \cap \Omega' = M$ such that π is parallel to π' and Ω is parallel to Ω'. The relation is reflexive and symmetric, and the transitivity follows from the transitivity of the parallelism in the set of generated affine planes and 4.9. Again analogously as in 4.9.4. [8] we can prove that each parallel class of blocks in B^α defines a partition of P and moreover if L and M are two $(0, \alpha + 1)$-sets which are parallel in a generated affine plane, then they also are parallel in B^α as defined above.

4.11. REMARKS. (1) Let π be a generated affine plane which does not contain the block L. Then π and L have no point in common iff L belongs to one of the parallel classes defined by the blocks of π.

(2) Parallel generated affine planes define the same $s + 2$ parallel classes of blocks.

(3) Let π and π' be generated affine planes such that two distinct intersecting blocks of π are parallel to two distinct intersecting blocks of π'. Then π is parallel to π'.

4.12. LEMMAS. *Suppose that* $s > 3$, *then the following hold.*
1. *Let* $y_1z_1u_1$ *and* $y_2z_2u_2$ *be two triangles of a generated affine plane* π, *which are in perspective from the point* w. *Suppose that all points* $w, y_1, z_1, u_1, y_2, z_2, u_2$ *are distinct and that all blocks* $wy_1, wz_1, wu_1, y_1z_1, z_1u_1, u_1y_1, y_2z_2, z_2u_2, u_2y_2$ *are distinct. If* y_1z_1, y_2z_2 *and if* z_1u_1, z_2u_2 *are parallel blocks of* B, *then also* u_1y_1 *and* u_2y_2 *are parallel.*
2. *Let* π *and* π' *be distinct parallel generated affine planes. Let* u *be a point not in* π *or* π' *and let* L_1, L_2 *be distinct blocks through* u *which do not belong to the parallel classes of blocks defined by* π *(or* π'*). Let* y_i *be the common point of* L_i *and* π, *and let* z_i *be the common point of* L_i *and* π'. *Then* $y_1y_2 \parallel z_1z_2$.

PROOFS. Completely analogous to the proofs of Lemmas 2.4.7. and 2.4.8. (due to J. A. Thas) in [7].

REMARK. As no proper partial geometries with $s = 3$ and $\alpha = 2$ can exist (see for instance [3]) the condition $s > 3$ is in fact superfluous.

5. CHARACTERIZATION OF $T_2^*(K)$

MAIN THEOREM. *Let* $S = (P, B, \mathrm{I})$ *be a net-inducible partial geometry with parameters satisfying* $t = \alpha(s + 2)$ *such that each triad* (x, y, z) *of points of* S *is contained in exactly* 0, 1 *or* $(t + 1)/(\alpha + 1)$ (> 1) *generated nets of order* $s + 1$ *and degree* $\alpha + 1$. *Then* $S \cong T_2^*(K)$.

PROOF. Since S is a net-inducible partial geometry we can define the set B^α of $(0, \alpha + 1)$-sets (see 4.3., 4.4.). We consider the 2-$((s + 1)^3, s + 1, 1)$ design S^α defined in 4.4. and we show that S^α is the design of points and lines of the affine space $\mathrm{AG}(3, s + 1)$.

(a) Any two points are contained in a unique block.
(b) Parallelism is an equivalence relation in the set of blocks of S^α. For each point x and each block L there exists a unique block M such that $x \in M$ and $L \parallel M$ (see 3.8 and 4.10).
(c) Suppose that $L \parallel M$, $L \neq M$, $x_1 \in L$, $x_2 \in M$. Let $w \in x_1x_2 \setminus \{x_1, x_2\}$ and let $x_1' \in L \setminus \{x_1\}$. We prove that $wx_1' \cap M \neq \emptyset$. Let π be a generated affine plane containing x_1x_1' but not w, and let π' be the generated affine plane parallel to π through x_2. Then the block M is contained in π' (see 4.11). In view of 4.12. $x_1x_1' \parallel x_2x_2''$ with x_2'' the common point of wx_1' and π'. Since M contains x_2 and $M \parallel x_1x_1'$, we have $M = x_2x_2''$. Hence M and wx_1' intersect.
(d) Each block has $s + 1$ (> 2) points.

By [11] S^α is the design of points and lines of an affine space. Since $|P| = (s + 1)^3$, the design S^α is the design of points and lines of AG(3, $s + 1$). Hence S is embedded in AG(3, $s + 1$). All the partial geometries embedded in affine spaces are known [13]. This yields that $S \cong T_2^*(K)$.

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